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ON THE LIMIT CYCLES AVAILABLE FROM POLYNOMIAL PERTURBATIONS OF THE BOGDANOV–TAKENS HAMILTONIAN*

BY

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ABSTRACT

The displacement map related to small polynomial perturbations of the planar Hamiltonian system $dH = 0$ is studied in the elliptic case $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$. An estimate of the number of isolated zeros for each of the successive Melnikov functions $M_k(h)$, $k = 1, 2, \ldots$ is given in terms of the order k and the maximal degree n of the perturbation. This sets up an upper bound to the number of limit cycles emerging from the periodic orbits of the Hamiltonian system under polynomial perturbations.

1. Introduction

We consider polynomial perturbations of the Hamiltonian vector field with a Hamiltonian $H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$:

$$\begin{aligned} \dot{x} &= y + \varepsilon f(x,y,\varepsilon), \\ \dot{y} &= -x + x^2 + \varepsilon g(x,y,\varepsilon). \end{aligned} \tag{1}$$

In (1), f and g are polynomials of x, y with coefficients depending analytically on the small parameter ε. The Hamiltonian H is known from the unfolding of a cusp singularity, called the Bogdanov–Takens unfolding. The unperturbed vector field has a periodic trajectory for Hamiltonian levels in $(0, \frac{1}{6})$.

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Let us denote $n = \max(\deg f, \deg g)$. We will assume that $n \geq 2$. Using the energy level $H = h$ as a parameter, $h \in (0, \frac{1}{6})$, we can express the first return mapping $\mathcal{P}$ of (1) in terms of h and ε. The corresponding displacement function $d(h,\varepsilon) = \mathcal{P}(h,\varepsilon) - h$ has a representation as a power series in ε,

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots, \tag{2}$$

which is convergent for small ε. The zeros in $(0, \frac{1}{6})$ of the first nonvanishing Melnikov function $M_k(h)$ in (2) determine the limit cycles in (1) emerging from periodic orbits of the unperturbed Hamiltonian system. In more general context, the problem (posed by V. Arnol'd [1]) of estimating the number of zeros of the Melnikov functions $M_k(h)$, given by Abelian integrals, is sometimes called the weakened (infinitesimal) 16th Hilbert problem.

If $M_1(h) \equiv 0$ in (2), then the perturbation is said to be conservative (but this might be misleading). For nonconservative perturbations, system (1) has been studied by Petrov [9], who found the exact estimate of $n-1$ for the number of zeros of $M_1(h)$ in $(0, \frac{1}{6})$, and by Mardešić [7], who gave the same sharp estimate for the total number of limit cycles in (1) bifurcating in the finite part of the (x,y)-plane. The "conservative" case was considered by Bao-yi Li and Zhi-fen Zhang [6]. Under the restriction $M_2(h) \not\equiv 0$, they obtained a sharp upper bound of $2n-2$ (n even) and $2n-3$ (n odd) for both the zeros of M_2 and the cycles of (1) in the finite plane. In this paper, we consider the problem without any restriction. We give a general estimate for any $M_k(h)$ and obtain the exact estimate of $3n-4$ for both the zeros and the cycles, provided $M_1(h) = M_2(h) \equiv 0$, $M_3(h) \not\equiv 0$. The general result is

THEOREM 1: *Assume that $M_k(h)$ is the first Melnikov function in (2) which does not vanish identically. Then $M_k(h)$ has no more than $k(n-1)$ zeros, counting the multiplicity.*

It turns out that the result in Theorem 1 is exact only for $k=1$ or $k=2$ and n even. For all other cases, the estimate can be improved by one.

THEOREM 2: *Assume that $M_k(h)$ is the first Melnikov function in (2) which does not vanish identically. If $k \geq 3$ or $k=2$ and n is odd, then $M_k(h)$ has no more than $k(n-1)-1$ zeros, counting the multiplicity.*

Apparently, this estimate may be sharp for the first few k only. There should exist a $k = k(n)$ after which the exact upper bound becomes stationary, see e.g. Proposition 4.1 in [2]. For instance, in the quadratic case $n=2$, it is well known that all Melnikov functions beginning from the second one M_2 can have at most two zeros [4], [10]. However, for $n \geq 3$, the level at which the number of zeros of M_k will stabilize, is unknown even as a hypothesis. In this connection, we prove

PROPOSITION 1: *For $k = 3$, the upper bound from Theorem 2 is sharp.*

Moreover, in Section 4 below we provide an example of a cubic perturbation with $M_1(h) = M_2(h) = M_3(h) \equiv 0$ and $M_4(h)$ having seven zeros. This fact suggests that the upper bound from Theorem 2 should be exact also for $k = 4$, $n \geq 3$.

It is a well-known fact [6], [7] that the upper bound for the number of zeros of the related Melnikov function in $(0, \frac{1}{6})$ also yields an estimate for the cycles. For a completeness, we formulate without proving a similar result. The proof is the same as in [6], [7].

THEOREM 3: *Assume that $M_k(h)$, $k \geq 3$, is the first nonvanishing Melnikov function in (2). Then system (1) has no more than $k(n-1) - 1$ limit cycles in the finite plane.*

We note that limit cycles escaping to infinity as $\varepsilon \to 0$ could appear in (1). They cannot be studied by inspecting the zeros of $M_k(h)$. For this reason we emphasize the fact that Theorem 3 concerns the limit cycles in the finite plane only.

In the table below, we sum the known results about exact upper bounds for both the cycles and zeros obtainable from kth order analysis (in ε) of an nth degree polynomially perturbed system (1):

Table 1

	$k = 1$	2	3	4	5	$\dots$
$n = 1$	0	0	0	0	0	$\dots$
2	1	2	2	2	2	$\dots$
3	2	3	5	7		$\dots$
4	3	6	8			$\dots$
5	4	7	11			$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\dots$
$2m$	$2m-1$	$4m-2$	$6m-4$			$\dots$
$2m+1$	$2m$	$4m-1$	$6m-1$			$\dots$

The paper is organized as follows. In the next Section 2 we consider the polynomial one-forms and their cohomology decompositions related to H (we accept the terminology from [3]). In Section 3 we use Françoise's recursive method [2], [5], [10], [11] to obtain an upper estimate for the degree of the polynomial one-form that, when integrated along the oval $\subset \{H=h\}$, yields $M_k(h)$. Based on this, we then prove Theorems 1 and 2. In Section 4 we provide, for $k=3$, an estimate from below for the number of zeros, needed to verify Proposition 1. Finally, we give an example suggesting that a similar result is true for $k=4$ as well.

2. Relative cohomology decompositions of one-forms

We consider polynomial one-forms of degree m,

$$\omega = g(x,y)dx - f(x,y)dy = \sum_{i+j\le m} b_{ij}x^iy^j\,dx - \sum_{i+j\le m} a_{ij}x^iy^j\,dy,$$

as well as polynomial one-forms of **weighted** degree m,

$$\begin{aligned}\omega =& g(x,y,H)dx - f(x,y,H)dy\\ =& \sum_{i+j+2k\le m} b_{ijk}x^iy^jH^k dx - \sum_{i+j+2k\le m} a_{ijk}x^iy^jH^k dy.\end{aligned}$$

For the polynomials $f(x,y,H)$ and $g(x,y,H)$ themselves we say they are of weighted degree m. In what follows, we will write for short w-deg$\,\omega = m$ and respectively w-deg$\,f=m$. The **leading part** of each polynomial (or one-form) of weighted degree m involves just the terms having weighted degrees exactly m.

Denote $\omega_{ij} = x^iy^j dx$, $\sigma_{ij} = x^iy^j dy$ and then put $\Omega_0 = \omega_{01} = ydx$, $\Omega_1 = \omega_{11} = xydx$,

$$J_k(h) = \int_{H=h} \Omega_k, \quad k=0,1, \quad h\in(0,\tfrac{1}{6}).$$

Below, $[s]$ denotes the entire part of s. We first prove

LEMMA 1 (Relative cohomology decomposition of one-forms): *Any polynomial one-form ω of degree m can be expressed as*

$$\omega = dR(x,y,H) + r(x,y,H)dH + \alpha(H)\Omega_0 + \beta(H)\Omega_1 \tag{3}$$

where $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$ and:

(i) *$R(x,y,H)$ and $r(x,y,H)$ are polynomials of weighted degrees $m+1$ and $m-1$ respectively.*

(ii) *For $x=0$, the leading part of r vanishes and the leading part of R does not depend on H.*

(iii) *$\alpha(h)$ and $\beta(h)$ are polynomials of degrees $[\frac{1}{2}(m-1)]$ and $[\frac{1}{2}(m-2)]$ respectively.*

Proof: The proof consists of straightforward calculations. Let $i+j \le m$. We have

$$\sigma_{ij} = \frac{1}{j+1} d(x^i y^{j+1}) - \frac{i}{j+1}\omega_{i-1,j+1},$$

so we only need to consider the forms ω_{ij}.

1) Assume first that $i \ge 2$. Then

$$\begin{aligned}\omega_{ij} =& x^i y^j\,dx = x^{i-2}y^j d\left(\frac{1}{3}x^3\right) = x^{i-2}y^j d\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - H\right)\\ =& x^{i-1}y^j\,dx + x^{i-2}y^{j+1}dy - x^{i-2}y^j\,dH\\ =& x^{i-1}y^j\,dx - \frac{i-2}{j+2}x^{i-3}y^{j+2}dx + d\left(\frac{1}{j+2}x^{i-2}y^{j+2}\right) - x^{i-2}y^j\,dH.\end{aligned}$$

Hence, it remains to consider the one-forms $y^j\,dx$ and $xy^j\,dx$.

2) Assume that j is even. Then

$$\begin{aligned}y^j =& (2H - x^2 + \frac{2}{3}x^3)^{j/2}\\ =& \sum_{k=0}^{j/2}(-1)^k\binom{j/2}{k}(2H)^{j/2-k}(x^2 - \frac{2}{3}x^3)^k\\ =& \sum_{k=0}^{j/2}\sum_{l=0}^{k}(-1)^{k+l}\binom{j/2}{k}\binom{k}{l}2^{j/2-k}(\frac{2}{3})^l H^{j/2-k}x^{2k+l}\\ =& \sum_{k=0}^{j/2}\sum_{l=0}^{k}C_{j,k,l}(\frac{2}{3})^l H^{j/2-k}x^{2k+l}.\end{aligned}$$

Hence

$$\begin{aligned}y^j\,dx =& \sum_{k=0}^{j/2}\sum_{l=0}^{k}\frac{C_{j,k,l}}{2k+l+1}(\frac{2}{3})^l H^{j/2-k}dx^{2k+l+1}\\ =& d\left(\sum_{k=0}^{j/2}\sum_{l=0}^{k}\frac{C_{j,k,l}}{2k+l+1}(\frac{2}{3})^l x^{2k+l+1}H^{j/2-k}\right)\\ &- \left(\sum_{k=0}^{j/2-1}\sum_{l=0}^{k}C_{j,k,l}\frac{j/2-k}{2k+l+1}(\frac{2}{3})^l x^{2k+l+1}H^{j/2-k-1}\right)dH\\ =& d\left(\sum_{k=0}^{j/2}\sum_{l=0}^{k}\frac{C_{j,k,l}}{2k+l+1}x^{2(k-l)+1}(x^2+y^2-2H)^l H^{j/2-k}\right)\\ &- \left(\sum_{k=0}^{j/2-1}\sum_{l=0}^{k}C_{j,k,l}\frac{j/2-k}{2k+l+1}x^{2(k-l)+1}(x^2+y^2-2H)^l H^{j/2-k-1}\right)dH.\end{aligned}$$

We proceed in a similar way with $xy^j dx$. Thus we have obtained that in the considered case the polynomials R and r in (3) take the required form.

3) Assume now that $j \geq 3$ is odd. Then we get

$$\begin{aligned} y^j dx =& y^{j-2}(2H - x^2 + \frac{2}{3}x^3)dx \\ =& 2Hy^{j-2}dx - y^{j-2}d\left(\frac{1}{3}x^3 - \frac{1}{6}x^4\right) \\ =& 2Hy^{j-2}dx - y^{j-2}d\left(\frac{1}{8}x^2 + \frac{1}{8}y^2 - \frac{1}{4}xy^2 - \frac{1}{4}H + \frac{1}{2}xH\right) \\ =& \frac{3}{2}Hy^{j-2}dx + \frac{j-2}{4j}y^j dx - \frac{1}{4}xy^{j-2}dx \\ & - \frac{1}{4j}d(1-2x)y^j + \frac{1}{4}(1-2x)y^{j-2}dH. \end{aligned}$$

After solving this equation with respect to $y^j dx$ one obtains

$$\begin{aligned} y^j dx = & \frac{6j}{3j+2}Hy^{j-2}dx - \frac{j}{3j+2}xy^{j-2}dx \\ & - \frac{1}{3j+2}d(1-2x)y^j + \frac{j}{3j+2}(1-2x)y^{j-2}dH. \end{aligned}$$

In a similar way we get

$$\begin{aligned} xy^j dx - \frac{1}{3j+4}y^j dx =& \frac{6j}{3j+4}Hxy^{j-2}dx - \frac{j}{3j+4}xy^{j-2}dx \\ & - \frac{1}{3j+4}d(1+x-2x^2)y^j \\ & + \frac{j}{3j+4}(1+x-2x^2)y^{j-2}dH. \end{aligned}$$

From the last two equations we get that, with some positive a_j, b_j, c_j,

$$\begin{aligned} y^j dx =& (a_j H^{(j-1)/2} + \text{l.o.t.})\Omega_0 + (\text{l.o.t.})\Omega_1 + dR_{j0} + r_{j0}dH, \\ xy^j dx =& (c_j H^{(j-1)/2} + \text{l.o.t.})\Omega_0 + (b_j H^{(j-1)/2} + \text{l.o.t.})\Omega_1 \\ & + dR_{j1} + r_{j1}dH, \end{aligned} \tag{4}$$

where R_{jk}, r_{jk} satisfy the requirements in (i), (ii) and by l.o.t. we have denoted the lower order terms in H.

By 1), 2) and 3), the proof of Lemma 1 is complete. ∎

Remark 1: The quotient vector space Ω_H of all polynomial one-forms $\omega = g(x,y)dx - f(x,y)dy$, modulo polynomial one-forms $dR + rdH$, was considered firstly by Petrov [8], without specifying the structure of R and r. Ω_H is a free module over the ring of polynomials $\mathbb{R}[h]$, under the multiplication $P(h)\tilde{\omega} = \widetilde{P(H)\omega}$ where $\tilde{\omega}$ denotes the equivalence class corresponding to ω.

LEMMA 2: *Assume that the polynomial one-form ω of degree m satisfies the identity $\int_{H=h} \omega \equiv 0$ for $h \in (0, \frac{1}{6})$. Then*

$$\omega = dR(x,y,H) + r(x,y,H)dH \tag{5}$$

where, in addition to (i) *and* (ii),

(iv) *The leading part of r is even with respect to y and has the same evenness with respect to x as $m-1$.*

(v) *The leading part of r is a constant multiplier of $\int_0^x my^{m-2}(s,H)ds$ (m even) and of $\int_0^x (m-1)sy^{m-3}(s,H)ds$ (m odd), where $y^2(s,H) = 2H - s^2 + \frac{2}{3}s^3$.*

Proof: By Lemma 1, we have

$$I(h) = \int_{H=h} \omega = \alpha(h)J_0(h) + \beta(h)J_1(h) \tag{6}$$

and $I(h) \equiv 0$ is equivalent to $\alpha(h) = \beta(h) \equiv 0$, see [6], [7], [9]. Hence, from part 3) of the proof of Lemma 1 above, all coefficients at $y^j dx$ and $xy^j dx$, j odd, will vanish. Then (iv) follows from part 2) of the preceding proof. To obtain (v), let us assume for definiteness that m is odd. Then the leading part of r comes from the one-form $\omega_{1,m-1} = xy^{m-1}dx$. Denoting $U(x,H) = \int_0^x sy^{m-1}(s,H)ds$, then $\omega_{1,m-1} = dU(x,H) - U_H(x,H)dH$ and, since $yy_H = 1$, (v) follows. ∎

COROLLARY 1: *Any polynomial one-form ω of weighted degree m can be decomposed into the form* (3) *where* (i), (iii) *still hold. If $\int_{H=h} \omega \equiv 0$, then* (5) *holds as well.*

COROLLARY 2: *Any integral $I(h) = \int_{H=h} \omega$ of polynomial one-form of weighted degree m has at most $m-1$ isolated zeros in $(0, \frac{1}{6})$.*

Proof: From Corollary 1, $I(h)$ can be expressed by (6). Then the well-known result of Petrov [9] applies to this integral which proves the assertion. ∎

COROLLARY 3: *Let ω be a polynomial one-form of odd weighted degree m with a leading part having a multiplier x^2. Then:*

(a) *$I(h) = \int_{H=h} \omega$ has at most $m-2$ isolated zeros in $(0, \frac{1}{6})$.*

(b) *In the decomposition (3), the leading part of r has a multiplier x^2.*

Proof: The leading part of ω is a linear combination of one-forms $x^{i+2}y^j H^k dx$, $x^{i+2}y^j H^k dy$, $2+i+j+2k=m$. The first form and the second one (when $i>0$) can be reduced to lower-degree forms, as we have done in point 1) of the proof of Lemma 1. It remains to consider $x^2 y^j H^k dy$. Since j is odd now, an integral of this form is zero. Therefore in (iii), $\alpha(h)$ and $\beta(h)$ are polynomials of degrees at most $[\frac{1}{2}(m-2)]$ and $[\frac{1}{2}(m-3)]$, respectively. From this observation, (a) follows. Further we have, modulo exact forms, that

$$x^2 y^j H^k dy = -\frac{k}{j+1} x^2 y^{j+1} H^{k-1} dH - \frac{2}{j+1} x y^{j+1} H^k dx,$$

henceforth (b) follows from point 2) in the proof of Lemma 1. ∎

3. Higher-order Melnikov functions and estimations of their zeros

To apply Françoise's procedure [2] for a calculation of the higher-order Melnikov functions $M_k(h)$, we write system (1) in a Pfaffian form

$$dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0 \tag{7}$$

where $\omega_j = g_j(x,y)dx - f_j(x,y)dy$ with $\deg f_j \le n$, $\deg g_j \le n$.

LEMMA 3 (Françoise's recursion formula, cf. [2], [5], [10]): *Assume that for some $k \ge 2$, $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$ in (2). Then*

$$M_k(h) = \int_{H=h} \Phi_k \tag{8}$$

where

$$\Phi_1 = \omega_1, \quad \Phi_m = \omega_m + \sum_{i+j=m} r_i\omega_j, \quad 2 \le m \le k, \tag{9}$$

and the functions r_i, $1 \le i \le k-1$ are determined successively from the representations $\Phi_i = dR_i + r_i dH$.

Proof: We obtain the proof by induction, cf. [2], [10]. The identity (3) yields that

$$\int_{H=h} \Phi_i = 0 \quad \Leftrightarrow \quad \Phi_i = dR_i + r_i dH.$$

We multiply (7) by $1+\varepsilon r_1+\cdots+\varepsilon^k r_k$ and then rearrange the monomials in the resulting expression to obtain

$$dH+\varepsilon(r_1dH-\omega_1)+\varepsilon^2(r_2dH-r_1\omega_1-\omega_2)+\cdots \\ +\varepsilon^k(r_kdH-r_{k-1}\omega_1-\cdots-r_1\omega_{k-1}-\omega_k)=O(\varepsilon^{k+1}).$$

By (9), this is equivalent to

$$dH=(\varepsilon dR_1+\varepsilon^2dR_2+\cdots+\varepsilon^{k-1}dR_{k-1})+\varepsilon^k(\Phi_k-r_kdH)+O(\varepsilon^{k+1}).$$

We integrate the last equation along the phase curve γ used in the determination of the first return map. Taking into account that

$$\int_\gamma dH=d(h,\varepsilon),\qquad \int_\gamma(\varepsilon dR_1+\varepsilon^2dR_2+\cdots+\varepsilon^{k-1}dR_{k-1})=O(\varepsilon^{k+1})$$

(the last estimate follows from the fact that $d(h,\varepsilon)=O(\varepsilon^k)$, cf. [5]), we obtain

$$d(h,\varepsilon)=\varepsilon^k\int_\gamma(\Phi_k-r_kdH)+O(\varepsilon^{k+1})=\varepsilon^k\int_{H=h}\Phi_k+O(\varepsilon^{k+1}).$$

Note that the above argument is also valid in a neighborhood of the saddle loop contained in $\{H=\frac{1}{6}\}$, cf. [4]. ∎

COROLLARY 4: *For any k, Φ_k is a polynomial one-form of weighted degree $kn-k+1$.*

Proof: From Lemma 1, $r_i(x,y,H)$ is a polynomial and $\text{w-deg}\, r_i=\text{w-deg}\,\Phi_i-1$. Then, by Lemma 3, we get recursively that $\text{w-deg}\,\Phi_1=n$, $\text{w-deg}\,\Phi_m=n+\max(\text{w-deg}\, r_i)=n-1+\text{w-deg}\,\Phi_{m-1}$ which yields the result. ∎

Proof of Theorem 1: From Corollaries 2 and 4, Theorem 1 follows. ∎

Proof of Theorem 2: (i) Let n be odd. Then, for any k, Φ_k is a form of odd weighted degree $m_k=k(n-1)+1$. Assume that $M_1(h)\equiv 0$. We write $r_1=\check r_1+\hat r_1$ where $\hat r_1$ and $\check r_1$ are the leading and the lower part of r_1 accordingly. This induces a corresponding decomposition $\Phi_2=\omega_2+r_1\omega_1=\check\Phi_2+\hat\Phi_2$ where $\text{w-deg}\,\check\Phi_2=2n-2$ and $\hat\Phi_2=\hat r_1\hat\omega_1$. From Lemmas 1, 2 and since $m_1=n$ is odd, we see $\hat r_1$ has x^2 as a multiplier. Then Corollary 3, applied to Φ_2, yields that $M_2(h)$ has at most $m_2-2=2n-3$ zeros and that $\hat r_2$ has x^2 as a multiplier. If $M_1(h)\equiv M_2(h)\equiv 0$, we repeat the above argument for $\Phi_3=\omega_3+r_1\omega_2+r_2\omega_1$, $\hat\Phi_3=\hat r_2\hat\omega_1$, and so on. Theorem 2 is proved for an odd n.

(ii) Assume that n is even. Then keeping the preceding notation, we prove the lemma below, from which Theorem 2 is a consequence. Indeed, $\hat{r}_3 = 0$ implies that $\hat{r}_k = 0$ for all $k \geq 3$, hence w-deg $\Phi_k = k(n-1)$, $k \geq 3$ and the result in the considered case follows from Corollary 2. Theorem 2 is completely proved. ∎

LEMMA 4: *Let n be even and $M_1(h) = M_2(h) \equiv 0$. Then $\hat{r}_3 = 0$.*

Proof: Recall that $\hat{r}_1$ and $\hat{R}_1$ are of odd weighted degrees $n-1$ and $n+1$, respectively. According to Lemma 2 (v) we have $\hat{r}_1 = b\int_0^x y^{n-2}(s,H)ds$. Then, modulo lower degree forms,

$$(10)\quad \hat{\Phi}_2 = \hat{r}_1\hat{\omega}_1 = \hat{r}_1(d\hat{R}_1 + \hat{r}_1 dH) = d(\hat{r}_1\hat{R}_1) + (\hat{r}_1^2 - \hat{R}_1\hat{r}_{1,H})dH - \hat{R}_1\hat{r}_{1,x}dx.$$

Note that by Lemma 2 (iv) $\hat{r}_1$, considered as a weighted polynomial, is an even function of y and an odd function of x. Writing $\hat{R}_1$ as a sum of an even and odd part with respect to y, $\hat{R}_1 = \hat{E}_1 + \hat{O}_1$, we see that $\hat{E}_1$ has x as a multiplier, while by Lemma 1 (ii), $\hat{O}_1 = ay^{n+1} + O(x^2)$. Then, $\hat{O}_1\hat{r}_{1,x}dx = aby^{2n-1}dx + \omega$ where ω is a form of an odd weighted degree $2n-1$ having a multiplier x^2. Therefore,

$$\int_{H=h}\Phi_2 = -ab\int_{H=h} y^{2n-1}dx + \alpha(h)J_0(h) + \beta(h)J_1(h)$$

where $\deg \alpha, \beta \leq n-2$. Since $M_2(h) \equiv 0$ this means that $ab = 0$, thus either $\hat{r}_1 = 0$ or $\hat{R}_1(0,y,H) = 0$. In the latter case, both $\hat{R}_1$ and $\hat{r}_1$ have a multiplier x, and moreover, $\hat{E}_1\hat{r}_{1,x}dx$ consists of one-forms $x^{1+i}y^jH^kdx$, i,j even. Using (10), Corollary 3 and the same argument as in the proof of Lemma 1, part 2), we see that $\hat{r}_2$ will have a multiplier x^2 in this case.

Now, if $\hat{r}_1 = 0$, then $\hat{r}_3 = 0$. Provided $\hat{R}_1(0,y,H) = 0$, then the one-form $y^n dy$ is missing in $\omega_1 = g_1(x,y)dx - f_1(x,y)dy$. Thus $\hat{r}_2\hat{\omega}_1$ will include the one-forms (of weighted degree $3n-2$) $x^{i+2}y^jH^kdx$, $x^{i+3}y^jH^kdy$, that are all reduced to lower-degree forms. Therefore $\hat{r}_3 = 0$ and the lemma is proved. ∎

4. Estimations from below

Proof of Proposition 1: It suffices to provide an example of a particular system for which $M_3(h)$ has just $3n-4$ zeros in $(0, \frac{1}{6})$. Assume first that $n > 2$ is even. To obtain the result in Table 1 for $k = 3$, we take in (7) $\omega_1 = d\{xA(y) + B(y)\} +$

$r_1(x,y)dH$, $\omega_2 = d\{xC(y)+D(y)\}$, $\omega_3 = \{xP(y)+Q(y)\}dx$ where $r_1 = xy^{n-3}$,

$$\begin{aligned} A(y) &= a_1y^3 + a_2y^5 + \cdots + a_{n/2-1}y^{n-1}, \\ B(y) &= b_1y^3 + b_2y^5 + \cdots + b_{n/2}y^{n+1}, \\ C(y) &= c_2y^4 + c_3y^6 + \cdots + c_{n/2}y^n, \\ D(y) &= d_2y^4 + d_3y^6 + \cdots + d_{n/2}y^n, \\ P(y) &= p_0y + p_1y^3 + \cdots + p_{n/2-1}y^{n-1}, \\ Q(y) &= q_0y + q_1y^3 + \cdots + q_{n/2-1}y^{n-1}. \end{aligned} \tag{11}$$

This choice implies that $M_1(h) = M_2(h) \equiv 0$, $M_3(h) = \int_{H=h}(\omega_3 + r_1\omega_2 + r_2\omega_1)$. First, we have

$$\begin{aligned} \int_{H=h} r_1\omega_2 &= \int_{H=h} \{x^2y^{n-3}C'(y) + xy^{n-3}D'(y)\}dy + xy^{n-3}C(y)dx \\ &= \int_{H=h} \{x[y^{n-3}C(y) - 2C_1(y)] - D_1(y)\}dx \end{aligned}$$

where $C_1' = y^{n-3}C'$, $D_1' = y^{n-3}D'$. Using (11), we get via direct calculations that

$$\int_{H=h} r_1\omega_2 = \int_{H=h} \left\{(\bar{c}_2x+\bar{d}_2)y^{n+1}+(\bar{c}_3x+\bar{d}_3)y^{n+3}+\cdots+(\bar{c}_{n/2}x+\bar{d}_{n/2})y^{2n-3}\right\}dx$$

with

$$\bar{c}_k = \frac{n-2k-3}{n+2k-3}c_k, \quad \bar{d}_k = -\frac{2k}{n+2k-3}d_k.$$

Second, to evaluate r_2, we obtain (modulo exact forms)

$$\begin{aligned} r_1\omega_1 =& xy^{n-3}\{A(y)dx + xA'(y)dy + B'(y)dy\} + x^2y^{2n-6}dH \\ =& -\{x[2A_1(y) - y^{n-3}A(y)] + B_1(y)\}dx + x^2y^{2n-6}dH, \end{aligned}$$

with $A_1' = y^{n-3}A'$, $B_1' = y^{n-3}B'$. We next take $y^2 = y^2(x,H) = 2H - x^2 + \frac{2}{3}x^3$ and denote

$$\begin{aligned} A_2(x,H) &= \int_0^x s[2A_1(y(s,H)) - y^{n-3}(s,H)A(y(s,H))]ds, \\ B_2(x,H) &= \int_0^x B_1(y(s,H))ds. \end{aligned}$$

Then $r_1\omega_1 = r_2(x,y,H)dH$ (modulo exact forms) where

$$r_2(x,y,H) = \partial_H A_2(x,H) + \partial_H B_2(x,H) + x^2y^{2n-6}.$$

Thus

$$\begin{aligned} \int_{H=h} r_2\omega_1 &= \int_{H=h} r_2 d\{xA(y) + B(y)\} + r_2r_1dH \\ &= -\int_{H=h} \{xA(y) + B(y)\}\{\partial_x r_2 dx + \partial_y r_2 dy + \partial_H r_2 dH\}. \end{aligned}$$

An easy calculation yields

$$\begin{aligned}\partial_x\partial_H A_2(x,H) =& x[2A_1'(y)-(n-3)y^{n-4}A(y)-y^{n-3}A'(y)]\partial_H y(x,H)\\ =& x[y^{n-4}A'(y)-(n-3)y^{n-5}A(y)].\end{aligned}$$

Similarly, $\partial_x\partial_H B_2(x,H)=y^{n-4}B'(y)$. We use all this to obtain

$$\int_{H=h} r_2\omega_1 = -(n-3)\int_{H=h} x^2\{3A(y)+2B(y)\}y^{2n-7}dy$$

$$-\int_{H=h}\{xA(y)+B(y)\}\{x[2y^{2n-6}+y^{n-4}A'(y)-(n-3)y^{n-5}A(y)]+y^{n-4}B'(y)\}dx.$$

Finally, we make use of (11) and, after long but elementary calculations, we obtain

$$\int_{H=h} r_2\omega_1 = -\int_{H=h}\left\{\sum_{k=0}^{\frac{n}{2}-1}(xv_k+w_k)y^{3n-2k-3}+\text{l.o.t.}\right\}dx.$$

In this formula, $v_0=0$, $w_0=(n+1)b_{n/2}^2$, and for $k=1,\dots,n/2-1$,

$$\begin{aligned}v_k =& \sum_{i+j=k}\left[(4-2j)a_{n/2-i}+(n-2k+5)b_{n/2-i}\right]a_{n/2-j}\\ &+\frac{12-4k}{3n-2k-3}a_{n/2-k+1}+\frac{2n-4k+6}{3n-2k-3}b_{n/2-k+1}\quad (a_{n/2}=0),\\ w_k =& \sum_{i+j=k}(n-2j+1)b_{n/2-i}b_{n/2-j}.\end{aligned}$$

After we express v_k and w_k, $k>0$, in the form

$$\begin{aligned}v_k =& \{(n-2k+5)b_{n/2}\}a_{n/2-k}+V_k(a_{n/2-k+1},b_{n/2-k+1},\dots,a_{n/2},b_{n/2}),\\ w_k =& \{(2n-2k+2)b_{n/2}\}b_{n/2-k}+W_k(b_{n/2-k+1},\dots,b_{n/2}),\end{aligned}$$

we see that they are all independently free constants, provided $b_{n/2}\neq 0$. Thus we get, with all constants independent,

$$\begin{aligned}M_3(h) =& \int_{H=h}(\omega_3+r_1\omega_2+r_2\omega_1)\\ =& \int_{H=h}\sum_{k=0}^{3n/2-3}\beta_k xy^{2k+1}dx+\int_{H=h}\sum_{k=0}^{3n/2-2}\alpha_k y^{2k+1}dx.\end{aligned}$$

The last formula guarantees, by (4), that $M_3(h)=\alpha(h)J_0(h)+\beta(h)J_1(h)$ where $\deg\alpha=\frac{3n}{2}-2$, $\deg\beta=\frac{3n}{2}-3$ and all of the $3n-3$ coefficients in α, β are independently free. Thus we can choose the coefficients in (11) so that M_3 will have just $3n-4$ zeros in $(0, \frac{1}{6})$. The proof for an even n is complete.

Consider now the case when $n > 2$ is odd.

We take $\omega_1 = d\{x^2A(y) + xB(y) + S(y)\} + r_1(y)dH$, $\omega_2 = d\{x^2C(y) + xD(y)\}$, $\omega_3 = \{xP(y) + Q(y)\}dx$ where $r_1 = y^{n-2}$,

$$\begin{aligned}
A(y) &= a_0y + a_1y^3 + \cdots + a_{(n-3)/2}y^{n-2},\\
B(y) &= b_0y + b_1y^3 + \cdots + b_{(n-1)/2}y^n,\\
S(y) &= s_0y + s_1y^3 + \cdots + s_{(n-1)/2}y^n,\\
C(y) &= c_1y^2 + c_2y^4 + \cdots + c_{(n-1)/2}y^{n-1},\\
D(y) &= d_1y^2 + d_2y^4 + \cdots + d_{(n-1)/2}y^{n-1},\\
P(y) &= p_0y + p_1y^3 + \cdots + p_{(n-3)/2}y^{n-2},\\
Q(y) &= q_0y + q_1y^3 + \cdots + q_{(n-3)/2}y^{n-2}.
\end{aligned}$$

The remaining part of the proof in this case repeats the argument stated above. The final formula we obtain for an odd n is

$$M_3(h) = \int_{H=h} (\omega_3 + r_1\omega_2 + r_2\omega_1) = \int_{H=h} \sum_{k=0}^{(3n-5)/2} (\beta_k xy^{2k+1} + \alpha_k y^{2k+1})dx.$$

We will omit the details. Proposition 1 is proved. ∎

Below we construct a cubic perturbation in (1) that has seven cycles, obtained from fourth-order analysis (in ε). We take in (7) $\omega_1 = dR_1(x, y, H) + r_1(x)dH$ where

$$R_1 = axy^3 + by^4 + c(\tfrac{1}{10}x^2y^2 - \tfrac{3}{5}x^2H - \tfrac{19}{20}x^4 + \tfrac{4}{3}x^3), \quad r_1 = c(x^2 - 2x).$$

The one-form ω_1 is so chosen (this is the most difficult part of the construction) that $r_1\omega_1 = r_2dH$, $r_2\omega_1 = \bar{r}_3dH$, $\bar{r}_3\omega_1 = \bar{r}_4dH + (\alpha_3H^3 + \cdots)\Omega_0 + (\beta_3H^3 + \cdots)\Omega_1$ (here and below, the equalities are assumed modulo exact forms). Next, one can take $\omega_2 = d(\alpha_2xy^2 + \beta_2x^2y^2)$, $\omega_3 = d(\beta_1y^3)$ and $\omega_4 = (\alpha_0y + \beta_0xy + \alpha_1y^3)dx$. As $r_1\omega_2 = \bar{\bar{r}}_3\,dH$ we obtain, with $r_3 = \bar{r}_3 + \bar{\bar{r}}_3$, that $\Phi_k = r_kdH$ for $k = 1, 2, 3$. Thus $M_k(h) \equiv 0$, $k = 1, 2, 3$. To calculate r_2 and r_3, we write $R_1 = axy^3 + R_0$ and consider R_0 as a function of x and H only (this is possible since $y^2 = 2H - x^2 + \frac{2}{3}x^3$). Then

$$\begin{aligned}
r_1(dR_0 + r_1dH) &= (r_1^2 + r_1R_{0H})dH + r_1R_{0x}dx\\
&= (r_1^2 + r_1R_{0H} - E_{1H})dH = r_0(x, H)dH
\end{aligned}$$

where $E_1(x, H) = \int_0^x r_1(s)R_{0s}(s, H)ds$. Since $(x^2 - 2x)d(xy^3) = 2y^3dH$, one obtains

$$r_2 = 2acy^3 + r_1^2 + r_1R_{0H} - E_{1H}.$$

Further, we have

$$\begin{aligned} r_2\omega_1 =&(2acy^3 + r_0)[d(axy^3 + R_0) + r_1 dH] \\ =&2acy^3 dR_0 + 2acy^3 r_1 dH + r_0 d(axy^3) \\ &+ 2acy^3 d(axy^3) + r_0 dR_0 + r_0 r_1 dH. \end{aligned}$$

After straightforward but long calculations, and making use of the formulas from point 3) in the proof of Lemma 1, one obtains

$$2acy^3 dR_0 + r_0 d(axy^3) = [8abc(2x^2 + y^2 - x^3)y^3 + \frac{6}{5}ac^2x^2y^3]dH.$$

Next, we have

$$\begin{aligned} r_0 dR_0 =& - R_0(r_{0H}dH + r_{0x}dx) \\ =& - R_0 r_{0H} dH - R_0(2r_1 r_1' + r_1' R_{0H} + r_1 R_{0xH} - E_{1xH})dx \\ =& - R_0 r_{0H} dH - R_0 r_1'(2r_1 + R_{0H})dx = (E_{2H} - R_0 r_{0H})dH, \end{aligned}$$

with $E_2(x, H) = \int_0^x R_0(s, H) r_1'(s)[2r_1(s) + R_{0H}(s, H)]ds$. In combination, all this yields

$$\begin{aligned} \bar{r}_3 =&8abc(2x^2 + y^2 - x^3)y^3 + 4ac^2(\frac{4}{5}x^2 - x)y^3 \\ &- E_{3H} + E_{2H} - R_0 r_{0H} + r_0 r_1, \\ \bar{\bar{r}}_3 =&c[\beta_2 x^4 + \frac{4}{3}(\alpha_2 - \beta_2)x^3 - 2\alpha_2 x^2], \end{aligned}$$

where $E_3(x, H) = a^2 c \int_0^x y^6(s, H)ds$. Writing $\bar{r}_3$ as a sum $\bar{r}_3 = e_3 + o_3$ of an even and odd function, we next have (modulo forms $dQ + qdH$)

$$\bar{r}_3\omega_1 = (e_3 + o_3)[d(axy^3 + R_0) + r_1 dH] = (-axy^3 e_{3x} + o_3 R_{0x})dx. \tag{12}$$

Using the functions we introduced, we get

$$\begin{aligned} e_{3x} =& - E_{3xH} + E_{2xH} - R_{0x} r_{0H} - R_0 r_{0xH} + r_{0x} r_1 + r_0 r_1' \\ =& - E_{3xH} + r_1' R_{0H}(2r_1 + R_{0H}) + r_1' R_0 R_{0HH} - R_{0x}(r_1 R_{0HH} - E_{1HH}) \\ &- R_0 r_1' R_{0HH} + r_1'(2r_1 + R_{0H})r_1 + (r_1^2 + r_1 R_{0H} - E_{1H})r_1' \\ =& - E_{3xH} - R_{0x}(r_1 R_{0HH} - E_{1HH}) + r_1'(3r_1^2 + 4r_1 R_{0H} + R_{0H}^2 - E_{1H}). \end{aligned}$$

Calculating

$$\begin{aligned} R_{0x} =&4by^2(x^2 - x) + c(\frac{3}{5}x^4 - \frac{2}{5}xy^2 - \frac{23}{5}x^3 + 4x^2), \\ R_{0H} =&4by^2 - \frac{2}{5}cx^2, \\ E_{1H} =&bc(\frac{8}{5}x^5 - 6x^4 + \frac{16}{3}x^3) - c^2(\frac{1}{5}x^4 - \frac{8}{15}x^3), \\ E_{3xH} =&6a^2cy^4 \end{aligned}$$

and replacing in (12), we obtain (this is a rather long and boring procedure which we omit here)

$$\int_{H=h} \bar{r}_3\omega_1 = ac\int_{H=h} (6a^2xy^7 - \tfrac{16}{5}bcy^7 + \tfrac{88}{25}c^2xy^5)dx.$$

A much more easy calculation yields

$$\begin{aligned}&\int_{H=h} (\omega_4 + r_1\omega_3 + r_2\omega_2 + \bar{\bar{r}}_3\omega_1)\\ &= \int_{H=h} [(\alpha_0 + \beta_0 x)y + (\alpha_1 + 2c\beta_1 - 2c\beta_1 x)y^3 + 2ac(\alpha_2 + 2\beta_2 x)y^5]dx.\end{aligned}$$

If we choose the coefficients α_k, β_k, a, b, c in a proper way, then clearly $M_4(h)$ will have seven zeros in $(0, \frac{1}{6})$. This completes our construction.

This last example suggests that the upper bound in Theorem 2 should be exact for $k = 4$, $n \geq 3$ as well.

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